

Quasi-periodicity in relative quasi-periodic tori

Francesco Fassò*, Luis C. García-Naranjo† and Andrea Giacobbe*

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Abstract

At variance from the cases of relative equilibria and relative periodic orbits of dynamical systems with symmetry, the dynamics in relative quasi-periodic tori (namely, subsets of the phase space that project to an invariant torus of the reduced system on which the flow is quasi-periodic) is not yet completely understood. Even in the simplest situation of a free action of a compact and abelian connected group, the dynamics in a relative quasi-periodic torus is not necessarily quasi-periodic. It is known that quasi-periodicity of the unreduced dynamics is related to the reducibility of the reconstruction equation, and sufficient conditions for it are virtually known only in a perturbation context. We provide a different, though equivalent, approach to this subject, based on the hypothesis of the existence of commuting, group-invariant lifts of a set of generators of the reduced torus. Under this hypothesis, which is shown to be equivalent to the reducibility of the reconstruction equation, we give a complete description of the structure of the relative quasi-periodic torus, which is a principal torus bundle whose fibers are tori of a dimension which exceeds that of the reduced torus by at most the rank of the group. The construction can always be done in such a way that these tori have minimal dimension and carry ergodic flow.

1 Introduction

1.1 Relative quasi-periodic tori

In a dynamical system with symmetry, a group-invariant subset of the phase space is called a *relative equilibrium* if its quotient with respect to the symmetry group G is an equilibrium of the reduced flow, and a *relative periodic orbit* if its quotient is a periodic orbit of the reduced flow. More generally, a relative set is a group-invariant subset whose quotient is invariant under the reduced dynamics.¹

The dynamics in relative equilibria and periodic orbits is well understood. It has been proved by Field [12, 13] and Krupa [17] that, under hypotheses which include the case of a free action of a compact and connected group, the flow in these two types of relative sets is conjugate to a linear flow on a torus (or, as we shall say, is ‘quasi-periodic’) of dimension not greater than the rank of the group (for relative equilibria) or of the rank of the group

*Università di Padova, Dipartimento di Matematica, Via Trieste 63, 35131 Padova, Italy. Email: fasso@math.unipd.it, giacobbe@math.unipd.it

†Departamento de Matemáticas y Mecánica, IIMAS-UNAM, Apdo Postal 20-726, Mexico City, 01000, Mexico. Email: luis@mym.iimas.unam.mx

¹Sometimes a different convention is used, and the terms relative equilibria and relative periodic orbits denote the individual orbits in such relative sets, rather than the relative sets themselves.

plus one (for relative periodic orbits). The dynamics in relative equilibria and periodic orbits in the case of a non-compact group, that need not be quasi-periodic, is well understood as well [2]. Details on these results, and their modifications under more general hypotheses, can be found in the quoted references and in the recent books [14, 8]. The related problem of reconstructing families of relative periodic orbits (e.g., when the entire reduced dynamics is periodic) has been studied in [15] in the case of a free action of a compact group (see also [9, 8]).

The situation is different if the reduced dynamics is quasi-periodic.

By a *relative quasi-periodic torus* we mean a relative set whose quotient is an invariant torus of the reduced system, that has dimension ≥ 2 and on which the reduced flow is conjugate to a linear flow. Clearly, it may be assumed that the reduced flow has nonresonant frequencies, so that the reduced invariant torus is the closure of any reduced trajectory on it (if not, restrict to a subtorus). It is well known that, even in the simplest case of a compact and connected group, the dynamics in a relative quasi-periodic torus need not be quasi-periodic. For a simple example see [24], where this failure is interpreted in terms of the appearance of ‘small divisors’ in the reconstruction process.

Per se, the specific topic of quasi-periodicity in relative quasi-periodic tori has so far received little attention. The only papers we know are [23, 24]. Reference [24] proves that, under natural hypotheses, relative quasi-periodic tori of Hamiltonian systems have quasi-periodic dynamics. Therefore, this problem is of interest only in the non-Hamiltonian case. Reference [23], even though considers only a special non-Hamiltonian case, lays down ideas that apply in general and links the problem to the theory of reducibility of quasi-periodic linear equations, a subject on which there is a vast literature (see e.g. [19] and references therein). Specifically, reference [23] proves that, under natural hypotheses that we will review in Section 3, the reducibility of the reconstruction equation implies the quasi-periodicity of the dynamics in the relative quasi-periodic torus.

However, reducibility of a quasi-periodic linear system is a difficult matter. There exist sufficient conditions of perturbative KAM type, but they require hypotheses (strong nonresonance and the presence of parameters that may be varied, see e.g. [3, 4]) that are hardly satisfied in the case of a single relative torus. More easily applicable to this case are results of approximate reducibility, called effective or almost reducibility (see e.g. [16, 6]), that require weaker conditions but give only closeness of the dynamics to quasi-periodicity for finite though long time scales [23].

At a deeper level, reference [23] proves that the reconstruction equation is reducible if and only if the unreduced vector field has an additional torus symmetry, a fact that will play a role in our approach as well.

1.2 Our results

The aim of this article is to introduce a new, though equivalent to reducibility, point of view on the subject of reconstruction of quasi-periodic dynamics. We focus on the simplest case of a free action of a compact and connected group and develop a reconstruction technique from reduced quasi-periodic dynamics that generalizes the known procedure used to reconstruct reduced periodic orbits, that is described in [13, 17, 15, 14, 8].

If the reduced flow is quasi-periodic with nonresonant frequencies, then no reduced orbit ever returns to the same point and the unreduced (or, as we shall say, the ‘reconstructed’) orbits never return to the same group orbit. This makes it impossible to define the basic object used in the reconstruction theory for periodic orbits—the group element associated to the first return map and variously called either ‘phase’ or ‘monodromy’ or ‘shift’.

We overcome this difficulty by applying the reconstruction techniques for periodic orbits not to the flow, but to a homology basis of the reduced torus. In order to do this, we need the assumption that, if the reduced torus is k -dimensional, a set of $k - 1$ of its generators lifts to a set of group-invariant vector fields that commute among each other and with the dynamics.

Under this hypothesis, we prove that the relative quasi-periodic torus has a structure analogous to that of relative equilibria and periodic orbits: it is fibered by invariant tori of a certain dimension $k + d$, with d between zero and the rank of the group, on which the flow is linear (Theorem 1). The frequencies of the reconstructed flow are the k ‘internal’ frequencies of the reduced flow plus d other ‘external’ frequencies that are identified in the reconstruction process in terms of the ‘phases’ of the lifts of the generators and of the Lie algebra of a certain torus in the group, whose construction depends on both geometric and dynamic properties of the system.²

Furthermore, as in the case of relative periodic orbits [12, 8], this construction can always be done in a way which is natural from the dynamical point of view as well, that is, the dimension $k + d$ of the reconstructed tori is ‘minimal’ and the flow is dense on them; this may require the consideration of a covering of the relative torus, and subharmonics of the reduced frequencies may appear (Theorem 2).

We also prove that our hypothesis is equivalent to the reducibility of the reconstruction equation (Theorem 3). Thus, our results on quasi-periodicity can be regarded as a different characterization of reducibility, with the addition, however, of a detailed description of the geometry of the relative quasi-periodic torus, which was missing so far.

1.3 Comments

Sample systems to which our technique may be applied can easily be constructed (see e.g. Section 3.2). It would be very interesting if our technique turned out to be applicable to some ‘natural’ problem, and in particular to nonholonomic systems, to which the well understood machinery of the Hamiltonian mechanics does not apply. Certainly, proving the existence of lifts with the desired properties may be difficult in practice but, as we will illustrate on some examples, it is not more difficult than proving reducibility.

Both, our hypotheses and the reducibility of the reconstruction equation, are sufficient conditions for quasi-periodicity of the dynamics. The reason why they are not also necessary is related to global aspects. The reconstruction equation, in its usual formulation, assumes that the relative set has the product structure $G \times \mathbb{T}^k$, but it is certainly possible that the dynamics be quasi-periodic in a relative torus which is a nontrivial bundle with fiber G and base \mathbb{T}^k . In our characterization, this triviality assumption is hidden in the fact that we assume the existence of global lifts of the generators, an assumption that might be weakened. We defer these generalizations to a future work.

Generalizations to non-connected groups and non-free actions can be treated easily, much in the same way as in the case of relative periodic orbits, and do not introduce important differences, see Section 2.2. The case of non-compact groups is significantly different, and would require a distinct analysis.

Finally, we note that a natural extension of the present study concerns the case in which the reduced dynamics has not just one relative quasi-periodic torus, but is integrable, in the sense that it has quasi-periodic dynamics. This is indeed the case considered in references [23, 24] when the reduced dynamics is quasi-periodic and in references [15, 9, 8] when the dynamics is periodic. Our technique is applicable to study such a case but, because of

²The terms ‘internal’ and ‘external’ frequencies are used in reducibility theory, see e.g. [19].

certain specificities of this problem—which are related to the global structure of the fibration by invariant tori—we prefer deferring this study to a separate work.

2 Reconstruction theorems

2.1 Statements

Consider a free action Ψ of a compact and connected Lie group G on a manifold M . Denote by \widehat{M} the quotient manifold M/G and by $\pi : M \rightarrow \widehat{M}$ the canonical projection. Let X be a G -invariant vector field on M . The reduced vector field is the vector field $\widehat{X} := \pi_* X$ on \widehat{M} .³

Assume also that the reduced vector field \widehat{X} has an invariant torus on which its flow is quasi-periodic. Specifically, there is an embedding

$$i : \mathbb{T}^k = (\mathbb{R}/\mathbb{Z})^k \rightarrow \widehat{M}$$

such that \widehat{X} is tangent to $\widehat{\mathcal{R}} := i(\mathbb{T}^k)$ and, denoting by $\varphi = (\varphi_1, \dots, \varphi_k)$ the angles in \mathbb{T}^k ,⁴

$$i^*(\widehat{X}|_{\widehat{\mathcal{R}}}) = \sum_{j=1}^k \omega_j \partial_{\varphi_j}$$

with $\omega_1, \dots, \omega_k \in \mathbb{R}$. We assume that the frequencies $\omega_1, \dots, \omega_k$ are nonresonant, that is, independent over \mathbb{Z} (if they are not, then the reconstruction applies to each nonresonant subtorus of $\widehat{\mathcal{R}}$). Hence, the closure of each reduced motion in $\widehat{\mathcal{R}}$ is the entire $\widehat{\mathcal{R}}$.

From now on we restrict our analysis to the *relative quasi-periodic torus*

$$\mathcal{R} := \pi^{-1}(\widehat{\mathcal{R}}),$$

which is a compact submanifold of M of dimension $k + \dim G$ and is G -invariant and X -invariant. The restriction $\pi|_{\mathcal{R}}$ of π to \mathcal{R} makes such a submanifold a G -principal bundle over $\widehat{\mathcal{R}}$. From now on we will write X for $X|_{\mathcal{R}}$, π for $\pi|_{\mathcal{R}}$ etc. (The manifold M will not play any role in the sequel; in fact, we might even have taken $M = \mathcal{R}$).

Recall that a dynamical symmetry of a vector field Y is any vector field that commutes with Y . Moreover, we say that a vector field S on \mathcal{R} is a *lift* of a vector field \widehat{S} on $\pi(\mathcal{R}) = \widehat{\mathcal{R}}$ if $\pi_* S = \widehat{S}$. As usual, we denote by $\mathcal{X}(\mathcal{R})$ the set of all vector fields on \mathcal{R} and by Φ_t^Y the time- t map of the flow of a vector field Y .

With a little abuse we denote by $\partial_{\varphi_1}, \dots, \partial_{\varphi_k}$ the vector fields $i_*(\partial_{\varphi_1}), \dots, i_*(\partial_{\varphi_k})$ in $\widehat{\mathcal{R}}$, and we call them *generators* of $\widehat{\mathcal{R}}$. Of course, the generators are not unique: any choice of angles on $\widehat{\mathcal{R}}$, namely of the embedding i , gives a set of them.

Theorem 1. *Under the stated hypotheses, assume that there exist lifts of $k - 1$ among a set of generators of $\widehat{\mathcal{R}}$ which are G -invariant, are dynamical symmetries of X , and pairwise commute. Then:*

- i. There exists a fibration of \mathcal{R} whose fibers are X -invariant manifolds diffeomorphic to \mathbb{T}^{k+d} , for a certain $0 \leq d \leq \text{rank } G$, on which the flow of X is conjugate to the linear flow on \mathbb{T}^{k+d} with frequencies $(\omega_1, \dots, \omega_k, \nu_1, \dots, \nu_d)$, with $(\nu_1, \dots, \nu_d) \in \mathbb{R}^d$ constant on \mathcal{R} .*

³The push-forward $\pi_* X$ of X is the vector field on \widehat{M} whose value in $\widehat{m} \in \widehat{M}$ is $\widehat{X}(\widehat{m}) = T_m \pi \cdot X(m)$ for any $m \in \pi^{-1}(\widehat{m})$.

⁴The pull-back $i^*(\widehat{X}|_{\widehat{\mathcal{R}}})$ is the unique vector field on \mathbb{T}^k such that $Ti \cdot i^*(\widehat{X}|_{\widehat{\mathcal{R}}}) = \widehat{X} \circ i$.

- ii. The base of this fibration of \mathcal{R} is diffeomorphic to G/T , where $T \subset G$ is a d -dimensional torus.

The above fibration, including the dimension d of the tori, is in general not unique. It will appear from the proof that, if $d < \text{rank } G$, then the same result is valid with tori of any dimension $d + 1, \dots, \text{rank } G$. In this statement it is not granted that the frequencies (ω, ν) are nonresonant, in which case there might exist fibrations of \mathcal{R} by tori of dimension smaller than such a d . It is however always possible to obtain a similar result with nonresonant frequencies, and hence tori of the minimal dimension and as such uniquely defined. In doing so, however, one realizes that the internal frequencies of the reconstructed motions might be subharmonics of those of the reduced motions:

Theorem 2. *Under the hypotheses of Theorem 1:*

- i. There exist integers $0 \leq d_0 \leq \text{rank } G$ and $r > 0$, a vector $\nu \in \mathbb{R}^{d_0}$, a fibration of \mathcal{R} whose fibers \mathcal{T}_m are X -invariant manifolds diffeomorphic to \mathbb{T}^{k+d_0} , and a covering map $c_m : \mathbb{T}^{k+d_0} \rightarrow \mathcal{T}_m$ that relates the linear flow on \mathbb{T}^{k+d_0} with frequencies $(\frac{\omega}{r}, \nu)$, which are nonresonant, to the flow of X in \mathcal{T}_m .
- ii. The base of this fibration of \mathcal{R} is diffeomorphic to G/T , where T is an abelian subgroup of G isomorphic to $T_0 \times F_0$, with T_0 a d_0 -dimensional torus of G and F_0 a finite abelian subgroup of G .

The proofs of the two theorems are given in Section 4.

2.2 Comments on Theorem 1

What Theorem 1 actually deals with is not the preimage under the canonical projection of a single quasi-periodic orbit, but of the entire invariant torus. This is why we prefer the expression ‘relative quasi-periodic torus’ to ‘relative quasi-periodic orbit’.

Theorem 1 may be regarded as an extension to the quasi-periodic case of the results for relative periodic orbits [13, 17, 15, 14, 8]. In the case of relative periodic orbits $k = 1$, and the hypotheses of the theorem yield no conditions and are hence always satisfied.

We remark that under the hypotheses of Theorem 1, there are not only $k - 1$, but k pairwise commuting, G -invariant dynamical symmetries of X that are lifts of the generators $\partial_{\varphi_1}, \dots, \partial_{\varphi_k}$. In fact, let S_1, \dots, S_{k-1} be lifts of, say, $\partial_{\varphi_1}, \dots, \partial_{\varphi_{k-1}}$, that have these properties and define

$$S_k = \frac{1}{\omega_k} \left(X - \sum_{j=1}^{k-1} \omega_j S_j \right); \quad (1)$$

then $\pi_* S_k = \partial_{\varphi_k}$ and S_1, \dots, S_k are lifts of $\partial_{\varphi_1}, \dots, \partial_{\varphi_k}$ with these same properties. In the proof, we will use such a set of k lifts.

Clearly, the lifts S_1, \dots, S_k define an action of \mathbb{R}^k that commutes with the action of G and leaves X invariant. To some extent, the proof of Theorem 1 consists in showing that this action is in fact an action of \mathbb{T}^k . Therefore, under the hypotheses of Theorem 1, the relative quasi-periodic torus \mathcal{R} is invariant under an action of the bigger group $\mathbb{T}^k \times G$. We will come back on this in Section 3.1 (Theorem 3) and in the Remark at the end of Section 4.3.

There are two rather natural generalizations of Theorem 1 to non-connected Lie groups and to non-free actions.

1. Non-connected Lie groups are of the form $G = H \rtimes K$, with H the connected component of the identity and K a finite group. If the hypotheses of Theorem 1 are satisfied, then they are also satisfied with the G -action replaced with that of its normal subgroup H .

The torus $\widehat{\mathcal{R}}$ embedded in M/G is covered, with fiber possibly a subgroup of K , by a torus embedded in M/H . The space \mathcal{R} is the preimage of this last torus, and the fibration of \mathcal{R} by tori can be obtained precisely with the same arguments.

2. If the G action is non-free, choose a point $m \in \mathcal{R}$ and let H be its stabilizer. Consider \mathcal{R}_H , the submanifold of all points of \mathcal{R} with orbit type H (see [1, pag. 16]). The vector field X and the lifts S_1, \dots, S_k described in Section 2.1 are tangent to \mathcal{R}_H , and the phases belong to $N(H)/H$ (where $N(H)$ is the normalizer of H). The hypotheses of Theorem 1 apply to the manifold \mathcal{R}_H under the free action of $N(H)/H$, and the theorem provides a foliation of \mathcal{R}_H in tori of dimension $k+d$ with d at most the rank of $N(H)/H$ (which is less than the rank of G). The foliation in tori can then be induced in the whole manifold \mathcal{R} using the G action.

Remark: The proof of Theorem 1 is trivial when G is abelian. A compact abelian group is a torus, hence $G \cong \mathbb{T}^d$, and one can use independent infinitesimal generators of the torus action as additional dynamical symmetries that commute with X and with S_1, \dots, S_k ; so, X is a \mathbb{T}^{k+d} -invariant vector field in \mathbb{T}^{k+d} .

2.3 Comments on Theorem 2

The reason why, in Theorem 2, the reconstructed motions may have the first k frequencies that are submultiples of those $\omega_1, \dots, \omega_k$ of the reduced motion can be understood on a simple example. The vector field

$$X = \partial_{\alpha_1} + \sqrt{2} \partial_{\alpha_2} + \frac{1}{2} \partial_{\alpha_3}$$

on $\mathcal{R} = \mathbb{T}^3 \ni (\alpha_1, \alpha_2, \alpha_3)$ is equivariant under the action of $G = S^1$ by translations of the third angle. The reduced system is the vector field $\widehat{X} = \partial_{\alpha_1} + \sqrt{2} \partial_{\alpha_2}$ on $\widehat{\mathcal{R}} = \mathbb{T}^2 \ni (\alpha_1, \alpha_2)$, so $k = 2$ and $(\omega_1, \omega_2) = (1, \sqrt{2})$. A possible reconstruction would obviously add back the third angle, leading to (a single) reconstructed torus of dimension 3, with internal frequencies $\omega = (1, \sqrt{2})$, $d = 1$ and external frequency $\nu = \frac{1}{2}$. But the dynamics of the unreduced system is dense in the 2-dimensional subtori of $\mathcal{R} = \mathbb{T}^3$ given by $\alpha_1 - 2\alpha_3 = \text{const}$. This suggests that it should also be possible to perform the reconstruction process so as to have $d_0 = 0$ and a fibration of $\mathcal{R} = \mathbb{T}^3$ by two-dimensional reconstructed tori, with no external frequencies. In fact, in the coordinates $\varphi_1 = \alpha_3$, $\varphi_2 = \alpha_2$, $\varphi_3 = \alpha_1 - 2\alpha_3$ on \mathcal{R} , the unreduced vector field is

$$X = \frac{1}{2} \partial_{\varphi_1} + \sqrt{2} \partial_{\varphi_2}$$

and its flow is conjugate to the linear flow on the two-dimensional tori $\varphi_3 = \text{const}$ with frequencies $(\frac{1}{2}, \sqrt{2}) = (\frac{\omega_1}{2}, \omega_2)$.

In Theorem 2, the dimension $k + d_0$ of the reconstructed torus may be anything between k and $k + \text{rank } G$. Mathematical examples of all these situations may of course be easily constructed. A very interesting example coming from nonholonomic mechanics is an n -dimensional generalization of the classical Veselova system [21, 22] considered in [10]. Such problem describes the motion of an n -dimensional rigid body with fixed point subject to a nonholonomic constraint. The phase space of the system is a vector subbundle \mathcal{D} of $TSO(n)$ of rank $n-1$ (and hence a manifold of dimension $n(n-1)/2 + (n-1)$), and the system has an $SO(n-1)$ symmetry-group. The reduced dynamics takes place in $\mathcal{D}/SO(n-1)$ (diffeomorphic to TS^{n-1}) and, up to a time parametrization, is quasi-periodic in $(n-1)$ -dimensional tori. Using techniques different from those used here, Fedorov and Jovanovic [10] prove that also the unreduced dynamics is, in the new time variable, quasi-periodic in tori of dimension $n-1$.

which foliate the phase space \mathcal{D} . If this case could be analyzed in the realm of our approach, complete resonance between the ‘internal’ and ‘external’ frequencies ω and ν would be found, making $d_0 = 0$.⁵

3 Comparison with reducibility

3.1 Reducibility and its equivalence to our hypotheses

We prove here the equivalence of our approach and of that based on the reducibility of the reconstruction equation, considered by Zenkov and Bloch [23]. We adapt the treatment in [23], that considers a specific problem⁶, to our general setting.

We consider a situation similar to that of Section 2.1, with a free action Ψ of a compact and connected Lie group G on a manifold \mathcal{R} and $\widehat{\mathcal{R}} = \mathcal{R}/G$ diffeomorphic to \mathbb{T}^k . As is typical in reducibility theory, we assume that $\mathcal{R} \rightarrow \mathcal{R}/G$ is a trivial G -bundle, so that there is an equivariant diffeomorphism from \mathcal{R} onto $\mathbb{T}^k \times G$. (In the sequel, ‘equivariant’ means always equivariant with respect to the G -action Ψ on \mathcal{R} and the G -action on $\mathbb{T}^k \times G$ by left translations on the factor G). Furthermore, we consider a G -invariant vector field on \mathcal{R} , whose reduced flow on $\widehat{\mathcal{R}}$ is conjugate to a linear flow on \mathbb{T}^k with frequencies $\omega = (\omega_1, \dots, \omega_k)$. Written in $\mathbb{T}^k \times G \ni (\varphi, g)$, the equations of motion of X on \mathcal{R} take the form of the linear skew-product system

$$\dot{\varphi} = \omega, \quad \dot{g} = g\xi(\varphi) \quad (2)$$

for some $\xi : \mathbb{T}^k \rightarrow \mathfrak{g}$, the Lie algebra of G . (For simplicity, we assume here that the group is a matrix group so that $L_g\xi = g\xi$). The partially integrated equation

$$\dot{g} = g\xi(t\omega) \quad (3)$$

is then called the reconstruction equation for X .

The reconstruction equation is said to be reducible if there exists a map $a : \mathbb{T}^k \rightarrow G$ such that the diffeomorphism

$$(\varphi, g) \mapsto (\varphi, ga(\varphi)^{-1}) =: (\varphi, h) \quad (4)$$

of $\mathbb{T}^k \times G$ onto itself conjugates system (2) to the system

$$\dot{\varphi} = \omega, \quad \dot{h} = h\rho \quad (5)$$

with a constant $\rho \in \mathfrak{g}$. This happens if and only if, writing for shortness $\frac{d}{dt}a(\varphi)$ for $\frac{\partial a}{\partial \varphi}(\varphi)\omega$, the map a is such that

$$a(\varphi)\xi(\varphi)a(\varphi)^{-1} - \left[\frac{d}{dt}a(\varphi)\right]a(\varphi)^{-1} \quad \text{is constant.} \quad (6)$$

Then ρ equals such a constant and the flow of equations (5) is quasi-periodic on tori of dimension at most $k + \text{rank } G$.

We remark that while the reconstruction equation is always reducible if the time dependence is periodic (Floquet theory), its reducibility is generally unknown in the quasi-periodic case (see e.g. [19] for a review of reducibility theory with emphasis on the quasi-periodic case).

Reference [23] proves that the reducibility of the reconstruction equation is equivalent to the invariance of the system under an extra \mathbb{T}^k -action. We use this fact to prove the equivalence between our hypotheses and reducibility.

⁵A personal communication with Y.N. Fedorov suggests that (part of) the frequencies of the unreduced motions are subharmonics of those of the reduced motions.

⁶A nonholonomic mechanical system with an $\text{SO}(n)$ symmetry, which is an n -dimensional generalization of the classical Suslov problem [20, 11] whose reduced dynamics is quasi-periodic

Theorem 3. *Assume that a vector field X on a compact manifold \mathcal{R} is invariant under a free action Ψ of a compact and connected Lie group G . Assume that $\widehat{\mathcal{R}} = \mathcal{R}/G$ is diffeomorphic to a k -dimensional torus and that the flow of the reduced vector field \widehat{X} on $\widehat{\mathcal{R}}$ is quasi-periodic with nonresonant frequencies. Then, the following three conditions are equivalent:*

- i. $\mathcal{R} \rightarrow \mathcal{R}/G$ is a trivial G -bundle and the reconstruction equation is reducible.
- ii. There exists a free action χ of \mathbb{T}^k on \mathcal{R} that leaves X invariant, commutes with Ψ and is such that the action (χ, Ψ) of $\mathbb{T}^k \times G$ on \mathcal{R} is free.
- iii. There exist G -invariant lifts of $k - 1$ among a set of generators of $\widehat{\mathcal{R}}$ that commute among themselves and with X .

The proof of this theorem uses some constructions from the proof of Theorem 1 and is postponed to Section 4.3.

Remarks: 1. Reference [23] proves, in a particular case, the equivalence of the two conditions i. and ii. of Theorem 3. However, some details are missing in that reference. In particular, the hypothesis of the triviality of the G -bundle \mathcal{R} is only implicit (without it, it is not possible to write the equations of motion in the form (2) and therefore the reconstruction equation (3)). Also, the freeness of the joint (χ, Ψ) action in condition ii. is not noticed.

2. In condition ii. of Theorem 3, the existence of an action of \mathbb{T}^k could be equivalently replaced by the existence of an action of \mathbb{T}^{k-1} .

3.2 Examples

We compare now our approach with reducibility in two examples.

Example 1. Consider the vector field

$$X = \partial_{\varphi_1} + \sqrt{2}\partial_{\varphi_2} + (f_1(\varphi_1) + f_2(\varphi_2))\xi_{\text{SO}(3)}(g)$$

on $\mathcal{R} = \mathbb{T}^2 \times \text{SO}(3) \ni (\varphi_1, \varphi_2, g)$, where $f_1, f_2 \in C^\infty(S^1)$ and $\xi \in \mathfrak{so}(3)$ ($\xi_{\text{SO}(3)}(g) \in T_g\text{SO}(3)$ is the value at g of the infinitesimal generator of ξ associated to left multiplication). The group $G = \text{SO}(3)$ acts by left multiplication on the $\text{SO}(3)$ factor of \mathcal{R} and leaves X invariant. The reduced space is $\widehat{\mathcal{R}} = \mathbb{T}^2$ and the reduced vector field

$$\widehat{X} = \partial_{\varphi_1} + \sqrt{2}\partial_{\varphi_2}$$

has quasi-periodic flow with frequencies $(\omega_1, \omega_2) = (1, \sqrt{2})$. The vector field

$$S_1 = \partial_{\varphi_1} + f_1\xi_{\text{SO}(3)}$$

is a lift of ∂_{φ_1} , is $\text{SO}(3)$ -invariant, and commutes with X . Therefore, given that $\text{SO}(3)$ has rank one, Theorem 1 (Theorem 2) implies that the flow of X on \mathcal{R} is conjugate to a (nonresonant) linear flow on tori of dimension either two or three (which possibility is realized depends on f_1, f_2, ξ).

To investigate the reducibility of the system write $f_1(\varphi_1) + f_2(\varphi_2) = \tilde{f}_1(\varphi_1) + \tilde{f}_2(\varphi_2) + \nu$ where \tilde{f}_1 and \tilde{f}_2 have zero averages and $\nu \in \mathbb{R}$ is the sum of the averages of f_1 and f_2 . The reconstruction equation is $\dot{g} = g\Omega(t, \sqrt{2}t)$ where

$$\Omega(\varphi_1, \varphi_2) = (\tilde{f}_1(\varphi_1) + \tilde{f}_2(\varphi_2) + \nu)\xi.$$

A fundamental matrix of the reconstruction equation is

$$A(t) = \exp\left(\xi \int_0^t \Omega(s) ds\right) = R(F_1(t) + 2^{-1/2}F_2(\sqrt{2}t) + \nu t)$$

where $F_1, F_2 \in C^\infty(S^1)$ are the primitives of \tilde{f}_1, \tilde{f}_2 that vanish at 0 and $R(x)$ is the rotation matrix $\exp(x\xi)$. Notice that $A(t)$ factors as $A(t) = \exp(t\nu\xi)a(t, \sqrt{2}t)$ where

$$a(\varphi_1, \varphi_2) = R(F_1(\varphi_1) + 2^{-1/2}F_2(\varphi_2))$$

satisfies the reducibility equation (6).

Example 2. Consider the vector field X on $M = \mathbb{T}^k \times \mathbb{T}^d \ni (\varphi, \vartheta)$

$$X = \sum_{i=1}^k \omega_i \partial_{\varphi_i} + \sum_{I=1}^d \xi_I(\varphi) \partial_{\vartheta_I}$$

with nonresonant $\omega_1, \dots, \omega_k$. X is invariant under the action of $G = \mathbb{T}^d$ on M by translations on the second factor. Our approach requires the existence of lifts

$$S_i = \partial_{\varphi_i} + \sum_{I=1}^d b_{iI}(\varphi) \partial_{\vartheta_I}, \quad i = 1, \dots, k,$$

such that

$$[S_i, S_j] = 0, \quad [S_i, X] = 0, \quad \forall i, j = 1, \dots, k. \quad (7)$$

Note that it is not restrictive to assume that the b_{iI} have zero averages. Conditions (7) are equivalent to

$$\frac{\partial b_{iI}}{\partial \varphi_i} - \frac{\partial b_{jI}}{\partial \varphi_j} = 0 \quad \forall i, j, I, \quad (8)$$

$$\frac{\partial \xi_I}{\partial \varphi_j} - \sum_{i=1}^k \omega_i \frac{\partial b_{iI}}{\partial \varphi_i} = 0 \quad \forall j, I. \quad (9)$$

In this case system (2) is $\dot{\varphi} = \omega$, $\dot{\vartheta} = \xi(\varphi)$ and, the group being abelian, the reducibility equation (6) is

$$\xi(\varphi) - \frac{\partial a}{\partial \varphi}(\varphi)\omega = \text{const} \quad (10)$$

and implies

$$\frac{\partial \xi_I}{\partial \varphi_i}(\varphi) - \sum_{j=1}^k \frac{\partial^2 a_I}{\partial \varphi_i \partial \varphi_j} \omega_j = 0 \quad \forall i, I.$$

This last equation coincides with (9) if $b_{iI} = \frac{\partial a_I}{\partial \varphi_i}$ and this set of b_{iI} satisfies the closure conditions (8).

Conversely, assume that our conditions are satisfied. Since the b_{iI} have zero average, equation (8) implies the existence of functions a_I such that $b_{iI} = \frac{\partial a_I}{\partial \varphi_i}$. Using the connectedness of the torus, integrating equation (9) gives the reducibility equation (10).

Remark: Proving the existence of solutions b_{iI} of equations (8), (9) is a different matter, that relies on the convergence of series that involve small denominators and depends on arithmetic properties of $\omega_1, \dots, \omega_k$ and on properties of the Fourier expansions of the ξ_I . This observation is in [24], who derives equation (9) in the simplest case $k = 1$, $d = 2$ as a condition for the integrability of X .

4 Proof of the theorems

4.1 Phases of commuting lifts of reduced periodic vector fields

First we recall the definition of phase for the periodic case, from [13, 15, 8]. We say that a G -invariant vector field $S \in \mathcal{X}(\mathcal{R})$ has *periodic reduced flow* if the reduced vector field $\hat{S} := \pi_*(S)$ has periodic flow with positive smooth period function $\hat{p} : \pi(\mathcal{R}) \rightarrow \mathbb{R}_+$; we call $p := \hat{p} \circ \pi$ the *lifted period* of S . The action of G on \mathcal{R} will be denoted by a dot.

If $S \in \mathcal{X}(\mathcal{R})$ has periodic reduced flow with lifted period p , then for any $m \in \mathcal{R}$, $\Phi_{p(m)}^S(m)$ belongs to the G -orbit of m and, given that the action is free,

$$\Phi_{p(m)}^S(m) = \gamma(m).m$$

for a unique element $\gamma(m) \in G$. This defines a map $\gamma : \mathcal{R} \rightarrow G$ that we call *phase* of S (monodromy and shift are also used). It is known that this map is smooth, has the G -equivariance property

$$\gamma(g.m) = g\gamma(m)g^{-1} \quad \text{for all } g \in G$$

and is constant along the flow of S : $\gamma(\Phi_t^S(m)) = \gamma(m)$ for all $t \in \mathbb{R}$ and $m \in \mathcal{R}$, see e.g. [9].

We now consider phases of commuting vector fields:

Lemma 1. *Let S and S' be two G -invariant commuting vector fields on \mathcal{R} with periodic reduced flows. Then:*

- i. *For each $m \in \mathcal{R}$, their phases $\gamma(m), \gamma'(m)$ commute.*
- ii. *The phase of S is constant along the flow of S' .*

Proof. i. If $m \in \mathcal{R}$, then

$$\Phi_{p'(m)}^{S'}(\Phi_{p(m)}^S(m)) = \Phi_{p'(m)}^{S'}(\gamma(m).m) = \gamma(m).\Phi_{p'(m)}^{S'}(m) = \gamma(m)\gamma'(m).m$$

where the second equality follows from the G -invariance of S' and the fact that the lifted period p' of S' is constant along G -orbits. Since the flows of S and S' commute we conclude that $\gamma(m)\gamma'(m).m = \gamma'(m)\gamma(m).m$; by the freeness of the action, $\gamma'(m)\gamma(m) = \gamma(m)\gamma'(m)$.

ii. Let $m' = \Phi_t^{S'}(m)$ for some $t \in \mathbb{R}$. Since the projected vector fields π_*S and π_*S' commute, the lifted period of S is constant along the flow of S' . Thus $p(m') = p(m)$ and $\gamma(m').m' = \Phi_{p(m')}^S(m') = \Phi_t^{S'}(\Phi_{p(m)}^S(m)) = \Phi_t^{S'}(\gamma(m).m) = \gamma(m).\Phi_t^{S'}(m) = \gamma(m).m'$, so that $\gamma(m') = \gamma(m)$. \square

4.2 Proof of Theorem 1

As noted in Section 2, see equation (1), under the hypotheses of Theorem 1 there exist G -invariant lifts $S_1, \dots, S_k \in \mathcal{X}(\mathcal{R})$ of $\partial_{\varphi_1}, \dots, \partial_{\varphi_k}$ which pairwise commute and are such that

$$X = \omega_1 S_1 + \dots + \omega_k S_k,$$

and hence commute with X as well. Thus

$$\Phi_t^X = \Phi_{t\omega_1}^{S_1} \circ \dots \circ \Phi_{t\omega_k}^{S_k} \quad \forall t \in \mathbb{R}.$$

We will write $\{S\}$ for the collection $\{S_1, \dots, S_k\}$ of the lifts and

$$\Phi_x^{\{S\}} := \Phi_{x_1}^{S_1} \circ \dots \circ \Phi_{x_k}^{S_k} \quad \forall x = (x_1, \dots, x_k) \in \mathbb{R}^k.$$

Thus $\Phi_t^X = \Phi_{\omega t}^{\{S\}}$, with $\omega = (\omega_1, \dots, \omega_k)$.

Fix a point $\overline{m} \in \mathcal{R}$. The vector fields S_1, \dots, S_k are G -invariant and have reduced periodic flows with unit period. Let $\overline{\gamma}_1, \dots, \overline{\gamma}_k$ be their phases at the point \overline{m} , $\overline{\gamma}_i = \gamma_i(\overline{m})$ if γ_i is the phase of S_i . Due to the commutativity of the phases, see Lemma 1, the set

$$\Gamma_2 := \{\overline{\gamma}_1^{z_1} \cdots \overline{\gamma}_k^{z_k} : (z_1, \dots, z_k) \in \mathbb{Z}^k\}$$

is an abelian subgroup of G . Let T_2 be a torus in G that contains the closure of Γ_2 . (Recall that a torus in a Lie group G is any connected compact abelian Lie subgroup of G ; all tori of G have dimension not greater than the rank of G , which is in fact the dimension of the ‘maximal tori’ [5, 14]). Let $\mathfrak{t}_2 \subseteq \mathfrak{g}$ be the Lie algebra of T_2 . Then, there exist vectors $\eta_1, \dots, \eta_k \in \mathfrak{t}_2$ such that $\exp(\eta_i) = \overline{\gamma}_i$ for $i = 1, \dots, k$. For economy of exposition, we will call these vectors ‘logarithms’ of the phases at \overline{m} ; they are not unique (being defined up to the addition of elements of \mathfrak{t}_2 that exponentiate to the group identity), so we make a choice of them. We will write $\eta = (\eta_1, \dots, \eta_k)$ and, if $x \in \mathbb{R}^k$, $x \star \eta$ for $x_1 \eta_1 + \dots + x_k \eta_k$.

Lemma 2. *The map⁷*

$$j : \mathbb{T}^k \times G \rightarrow \mathcal{R}, \quad j(\alpha, g) = g \exp(-\alpha \star \eta) \cdot \Phi_\alpha^{\{S\}}(\overline{m}) \quad (11)$$

is a diffeomorphism.

Proof. The map j is well defined because $j(\alpha + z, g) = j(\alpha, g)$ for all $z \in \mathbb{Z}^k$. In fact

$$j(\alpha + z, g) = g \exp(-\alpha \star \eta) \exp(-z \star \eta) \cdot \Phi_\alpha^{\{S\}}(\Phi_z^{\{S\}}(\overline{m})) = j(\alpha, g)$$

given that, by the G -invariance of the lifts,

$$\Phi_\alpha^{\{S\}}(\Phi_z^{\{S\}}(\overline{m})) = \Phi_\alpha^{\{S\}}(\overline{\gamma}_1^{z_1} \cdots \overline{\gamma}_k^{z_k} \cdot \overline{m}) = \exp(z \star \eta) \cdot \Phi_\alpha^{\{S\}}(\overline{m}).$$

To prove that j is a diffeomorphism it suffices to show that it is a local diffeomorphism at each point, and that it is bijective.

To prove that j is a local diffeomorphism, consider a vector $(a, \xi) \in \mathbb{R}^k \times \mathfrak{g}$. Let $v \in T_{(\alpha, g)}(\mathbb{T}^k \times G)$ be the vector which is tangent at $t = 0$ to the curve $t \mapsto (\alpha + ta, \exp(t\xi)g)$. Since the image via j of this curve is

$$t \mapsto \exp(t\xi)g \exp(-(\alpha + at) \star \eta) \cdot \Phi_{\alpha+ta}^{\{S\}}(\overline{m}),$$

by differentiation one finds

$$T_{(\alpha, g)}j \cdot v = X_\xi(j(\alpha, g)) - \sum_i a_i X_{\eta_i}(j(\alpha, g)) + \sum_i a_i S_i(j(\alpha, g))$$

where X_ξ, X_{η_i} are the infinitesimal generators of the group action associated to the elements $\xi, \eta_i \in \mathfrak{g}$. Since the S_i ’s are not tangent to the orbits of G , the vanishing of $T_{(\alpha, g)}j \cdot v$ requires $a = 0$ and hence the vanishing of X_ξ ; thus $\xi = 0$.

To prove injectivity assume that $j(\alpha, g) = j(\beta, h)$. Then

$$\Phi_{\alpha-\beta}^{\{S\}}(\overline{m}) = \exp(\alpha \star \eta) g^{-1} h \exp(-\beta \star \eta) \cdot \overline{m} \quad (12)$$

and $\Phi_{\alpha-\beta}^{\{S\}}(\overline{m})$ must be in the G -orbit of \overline{m} . This happens if and only if $\alpha - \beta \in \mathbb{Z}^k$. Hence $\Phi_{\alpha-\beta}^{\{S\}}(\overline{m}) = \overline{\gamma}_1^{\alpha_1 - \beta_1} \cdots \overline{\gamma}_k^{\alpha_k - \beta_k} \cdot \overline{m} = \exp((\alpha - \beta) \star \eta) \cdot \overline{m} = \exp(\alpha \star \eta) \exp(-\beta \star \eta) \cdot \overline{m}$ and

⁷ To keep the notation simple, we ordinarily avoid distinguishing between a point $\alpha \in \mathbb{R}^k$ and the equivalence class $\alpha(\text{mod } 1) \in \mathbb{T}^k$. For instance, here, the right hand side of the definition of the map j is a function on $\mathbb{R}^k \times G$; the correctness of this definition is checked in the proof.

equality (12) reduces to $\exp(\alpha \star \eta) \cdot (\exp(-\beta \star \eta) \cdot \overline{m}) = \exp(\alpha \star \eta) g^{-1} h \cdot (\exp(-\beta \star \eta) \cdot \overline{m})$, which gives $g^{-1} h = e_G$, the identity of the group. Thus $\alpha \equiv \beta \pmod{1}$ and $g = h$.

Finally, for any $m \in \mathcal{R}$ there exist $\beta \in \mathbb{T}^k$ such that $\Phi_\beta^{\{\hat{S}\}}(\pi(m)) = \pi(\overline{m})$. Hence $\Phi_\beta^{\{\hat{S}\}}(\pi(m)) = h \cdot \overline{m}$ for some $h \in G$ and $m = j(\beta, h \exp(\beta \star \eta))$. Thus, j is surjective. \square

Note now that $\overline{m} = j(0, e_G)$, where e_G is the identity of G , and

$$\Phi_t^X(j(\alpha, g)) = j(\alpha + t\omega, g \exp(t\omega \star \eta)) \quad \forall t, \alpha, g. \quad (13)$$

The one-parameter subgroup

$$\Gamma_1 := \{\exp(t\omega \star \eta) : t \in \mathbb{R}\}$$

of G is an abelian and connected subgroup of T_2 . Hence, its closure is a torus $T_1 \subseteq T_2$ of some dimension $d_1 \leq d_2$.

The set

$$\mathcal{T}_{\overline{m}} := j(\mathbb{T}^k \times T_1)$$

is diffeomorphic to $\mathbb{T}^k \times T_1$ and is invariant under the flow of X : from (13), if $(\beta, h) \in \mathbb{T}^k \times T_1$ then $\Phi_t^X(j(\beta, h)) = j(\beta + \omega t, h \exp(t\omega \star \eta)) \in \mathcal{T}_{\overline{m}}$ because both h and $\exp(t\omega \star \eta)$ belong to the subgroup T_1 . If $m \in \mathcal{R}$ then $m = j(\beta, g)$ for unique $\beta \in \mathbb{T}^k$, $g \in G$ and we define

$$\mathcal{T}_m := g \cdot \mathcal{T}_{\overline{m}}.$$

Lemma 3.

- i. For any $m \in \mathcal{R}$, \mathcal{T}_m is diffeomorphic to $\mathbb{T}^k \times T_1$ and is X -invariant.
- ii. For any $m, m' \in \mathcal{R}$, the sets \mathcal{T}_m and $\mathcal{T}_{m'}$ are either equal or disjoint.
- iii. The sets \mathcal{T}_m , $m \in \mathcal{R}$, are the fibers of a fibration of \mathcal{R} with base diffeomorphic to G/T_1 .

Proof. i. Since $g \cdot j(\beta, h) = gh \exp(-\beta \star \eta) \cdot \Phi_\beta^{\{\hat{S}\}}(\overline{m}) = j(\beta, gh)$,

$$\mathcal{T}_{g \cdot \overline{m}} = j(\mathbb{T}^k \times gT_1).$$

The X -invariance of $g \cdot \mathcal{T}_{\overline{m}}$ follows from the X -invariance of $\mathcal{T}_{\overline{m}}$ and from the G -invariance of X : if $(\beta, h) \in \mathbb{T}^k \times T_1$ then $\Phi_t^X(g \cdot j(\beta, h)) = g \cdot \Phi_t^X(j(\beta, h)) \in g \cdot \mathcal{T}_{\overline{m}}$.

ii. Since $\mathcal{T}_{g \cdot \overline{m}} = j(\mathbb{T}^k \times gT_1)$ this follows from the facts that j is a diffeomorphism of $\mathbb{T}^k \times G$ onto \mathcal{R} and that lateral classes in a group either coincide or are disjoint.

iii. This follows from the fact that the sets $\mathbb{T}^k \times gT_1$ are the fibers of a fibration of $\mathbb{T}^k \times G$ with base G/T_1 . \square

We now prove that the restriction of the flow of X to each set \mathcal{T}_m , $m \in \mathcal{R}$, is conjugate to one and the same linear flow on \mathbb{T}^{k+d_1} , where d_1 is the dimension of T_1 . Choose an integral basis $\xi = \{\xi_1, \dots, \xi_{d_1}\}$ of the Lie algebra $\mathfrak{t}_1 \subseteq \mathfrak{g}$ of T_1 ; ‘integral basis’ means that it generates the lattice of elements that exponentiate to the unity: for $\zeta \in \mathfrak{t}_1$, $\exp \zeta = e_G$ if and only if $\zeta = \sum_i z_i \xi_i$ with all $z_i \in \mathbb{Z}$. Then, $\omega \star \eta = \sum_{i=1}^{d_1} \nu_i \xi_i$ or

$$\omega \star \eta = \nu \star \xi \quad (14)$$

for some $\nu = (\nu_1, \dots, \nu_{d_1})$. Moreover, for any $g \in G$, the map

$$i_g := \mathbb{T}^k \times \mathbb{T}^{d_1} \rightarrow \mathcal{T}_{g \cdot \overline{m}}, \quad i_g(\alpha, \beta) = j(\alpha, g \exp(\beta \star \xi))$$

is a diffeomorphism.

Lemma 4. *For any $g \in G$, i_g conjugates the linear flow*

$$(t, (\alpha, \beta)) \mapsto (\alpha + t\omega, \beta + t\nu) \pmod{1}$$

on $\mathbb{T}^k \times \mathbb{T}^{d_1}$ to the restriction of the flow of X to $\mathcal{T}_{g, \overline{m}}$.

Proof. $\Phi_t^X(i_g(\alpha, \beta)) = \Phi_t^X(j(\alpha, g \exp(\beta \star \xi))) = j(\alpha + t\omega, g \exp(\beta \star \xi + t\omega \star \eta))$. This equals $i_g(\alpha + t\omega, \beta + t\nu)$ given that $\omega \star \eta = \nu \star \xi$. \square

This completes the proof of Theorem 1, with $d = d_1$ and $T = T_1$.

Remarks: 1. The external frequencies ν are related to the internal ones by a linear relation, see (14). This might be seen as related to the linearity of the reconstruction equation.

2. The choice of the lifts of the generators, of their phases and obviously of the point \overline{m} are not unique. The entire construction depends on them. Only the invariant tori of \mathcal{R} of *minimal* dimension, being uniquely defined by the dynamics as closure of trajectories, are independent of these choices.

3. What we need in the proof of Theorem 1 are the logarithms η_i of the phases $\overline{\gamma}_i$ and the fact that they commute. This implies that they belong to some abelian Lie subalgebra of \mathfrak{g} , and the torus T_2 is used only to define such a subalgebra. The dimension of T_2 is immaterial to our construction, and even if it might seem natural to choose it as small as possible, there is no need to do that within the proof of Theorem 1. The reason is that we might need to change anyway this choice in the proof of Theorem 2, to remove resonances between the internal and the external frequencies.

4.3 Proof of Theorem 3

i. \implies ii. Under hypothesis i., there exists an equivariant diffeomorphism D_1 between \mathcal{R} and $\mathbb{T}^k \times G$. ('Equivariant' has here the same meaning as in Section 3.1). By the reducibility of the reconstruction equation there is a diffeomorphism $D_2 : \mathcal{R} \rightarrow \mathbb{T}^k \times G$ that conjugates the vector field X to the system (5) on $\mathbb{T}^k \times G$, namely

$$\dot{\varphi} = \omega, \quad \dot{h} = h\rho \tag{15}$$

with a constant $\rho \in \mathfrak{g}$. The diffeomorphism D_2 is the composition of D_1 and of the diffeomorphism $(\varphi, g) \mapsto (\varphi, ga(\varphi)^{-1})$ as in (4), and is thus equivariant. System (15) is invariant under the action Λ of $\mathbb{T}^k \times G$ on itself by left translations,

$$\Lambda_{(\alpha, g)}(\beta, h) = (\alpha + \beta, gh).$$

Therefore, X is invariant under the action $\tilde{\Lambda}$ of $\mathbb{T}^k \times G$ on \mathcal{R} given by

$$\tilde{\Lambda}_{(\alpha, g)} := D_2^{-1} \circ \Lambda_{(\alpha, g)} \circ D_2$$

and this action is free because Λ is free. By the equivariance of D_2 , the restriction of the action $\tilde{\Lambda}$ to the subgroup $\{0\} \times G$ coincides with the action Ψ of G on \mathcal{R} . Hence, the restriction of $\tilde{\Lambda}$ to the subgroup $\mathbb{T}^k \times \{e_G\}$ gives an action χ of \mathbb{T}^k on \mathcal{R} that has the properties of condition ii. of Theorem 3.

ii. \implies iii. Let $Y_1, \dots, Y_k \in \mathcal{X}(\mathcal{R})$ be a set of infinitesimal generators of the action χ of \mathbb{T}^k on \mathcal{R} . Since χ commutes with Ψ , the vector fields Y_i are π -related to vector fields \hat{Y}_i in $\tilde{\mathcal{R}}$ (with, of course, $\pi : \mathcal{R} \rightarrow \mathcal{R}/G$ the canonical projection). The χ -invariance of X and the density of its trajectories on \mathcal{R}/G implies that the vector fields \hat{Y}_i have the form $\sum_{j=1}^k r_{ij} \partial_{\varphi_j}$

with constant r_{ij} . The freeness hypothesis implies that the vector fields \widehat{Y}_i are independent. Hence, there exist constant coefficients s_{ij} such that, for every i , $\partial_{\varphi_i} = \sum_{j=1}^k s_{ij} \widehat{Y}_j$. The $k-1$ vector fields $S_i = \sum_{j=1}^k s_{ij} Y_j$, $i = 1, \dots, k-1$, have the properties of condition iii.

iii. \implies i. Under hypothesis iii., the construction done in the proof of Theorem 1 is valid. In particular, by Lemma 2, \mathcal{R} is diffeomorphic the map $j : \mathbb{T}^k \times G \rightarrow \mathcal{R}$ as in (11) is an equivariant diffeomorphism, and hence \mathcal{R} is a trivial G -bundle. Formula (13) shows that j conjugates X to the constant system

$$\dot{\varphi} = \omega, \quad \dot{g} = g \omega \star \eta$$

on $\mathbb{T}^k \times G$. Hence, the reconstruction equation is reducible (with $\rho = \omega \star \eta$).

Remark: In the hypotheses of Theorem 1, the additional action χ of \mathbb{T}^k is constructed as follows. Consider the free action J of $\mathbb{T}^k \times G$ on \mathcal{R} given by

$$J_{(\alpha, g)} := j \circ \Lambda_{(\alpha, g)} \circ j^{-1}$$

From

$$J_{(\alpha, g)}(j(\beta, h)) = j(\alpha + \beta, gh)$$

and (13) it follows that $\Phi_t^X(J_{(\alpha, g)}(j(\beta, h))) = j(\alpha + \beta + \omega t, gh \exp(t\omega \star \eta)) = J_{(\alpha, g)}(\Phi_t^X(j(\beta, h)))$, which shows that X is J -invariant. Moreover, $J|_{\{0\} \times G} = \Psi$ and $\chi := J|_{\mathbb{T}^k \times \{e_G\}}$ is an action of \mathbb{T}^k on \mathcal{R} with all the properties of condition ii.

4.4 Proof of Theorem 2

In the proof of Theorem 2 we use the entire construction done in the proof of Theorem 1. If the frequency vector (ω, ν) constructed in that proof happens to be nonresonant then Theorem 2 is valid with $d_0 = d_1$, $T = T_1$ and $r = 1$. It remains to be considered the case in which the vector (ω, ν) is resonant.

A resonance of $(\omega, \nu) \in \mathbb{R}^{k+d_1}$ is a nonzero integer vector $(p, q) \in \mathbb{Z}^{k+d_1}$ such that $p \cdot \omega + q \cdot \nu = 0$. Resonances of (ω, ν) form a lattice Λ of \mathbb{Z}^{k+d_1} of rank $l \geq 1$. We recall that a lattice of \mathbb{Z}^m of rank l is the set of all linear combinations with integer coefficients of l linearly independent vectors of \mathbb{Z}^m , called a basis of the lattice. In our case, let

$$(\tilde{p}^1, \tilde{q}^1), \dots, (\tilde{p}^l, \tilde{q}^l) \in \mathbb{Z}^{k+d_1}$$

be a basis of the lattice Λ .

Remember that the vector $\omega \in \mathbb{R}^k$ is nonresonant by assumption. The density of the set $\{\exp(t\omega \star \eta) = \exp(t\nu \star \xi) : t \in \mathbb{R}\}$ in the d_1 -dimensional torus T_1 implies that the vector $\nu \in \mathbb{R}^{d_1}$ is nonresonant as well. Thus, each of the two groups of vectors $\tilde{p}^1, \dots, \tilde{p}^l \in \mathbb{Z}^k$ and $\tilde{q}^1, \dots, \tilde{q}^l \in \mathbb{Z}^{d_1}$ forming the basis of Λ is independent over \mathbb{Z} . (For, if $\sum_i c_i \tilde{p}^i = 0$ with $c_1, \dots, c_l \in \mathbb{Z}$, then $0 = \sum_i c_i (\tilde{p}^i \cdot \omega + \tilde{q}^i \cdot \nu) = \sum_i c_i \tilde{q}^i \cdot \nu$ and hence $\sum_i c_i \tilde{q}^i = 0$; but then $\sum_i c_i (\tilde{q}^i, \tilde{p}^i) = 0$ and all $c_i = 0$). From this it follows that $l \leq \min(k, d_1)$. Moreover, it is possible to put the resonances in a simpler form:

Lemma 5. *There is an integral basis⁸ $\xi' = \{\xi'_1, \dots, \xi'_{d_1}\}$ of \mathfrak{t}_1 such that, if $\nu' \in \mathbb{R}^{d_1}$ denotes the components of $\nu \star \xi \in \mathfrak{t}_1$ in this basis (that is, $\nu \star \xi = \nu' \star \xi'$), then the resonant lattice $\Lambda' \subset \mathbb{Z}^{k+d_1}$ of $(\omega, \nu') \in \mathbb{R}^{k+d_1}$ has a basis formed by l vectors*

$$(p^1, r_1 e^1), \dots, (p^l, r_l e^l) \in \mathbb{Z}^{k+d_1} \tag{16}$$

⁸The notion of integral basis is defined just after Lemma 3.

where $p^1, \dots, p^l \in \mathbb{Z}^k$, e^i denotes the i -th vector of the standard basis of \mathbb{Z}^{d_1} , and the r_i 's are positive integers with the property that if $r_i < r_j$ then r_i divides r_j .

Proof. Let Q be the $d_1 \times l$ integer matrix with columns $\tilde{q}^1, \dots, \tilde{q}^l$. By the Smith Normal Form Theorem (see e.g. [7]), there exist a $d_1 \times d_1$ integer matrix Z and an $l \times l$ integer matrix C , both invertible over \mathbb{Z} (that is, unimodular, or having determinant ± 1), such that the matrix ZQC^T has the block structure

$$ZQC^T = \begin{pmatrix} \text{diag}(r_1, \dots, r_l) \\ O_{d_1-l, l} \end{pmatrix} \quad (17)$$

where $O_{d_1-l, l}$ is the $(d_1 - l) \times l$ null block and r_1, \dots, r_l are nonnegative integers such that if $0 \neq r_i < r_j$ then r_i divides r_j . Since Q has rank l , all $r_j \neq 0$. Let $Z_{ij} \in \mathbb{Z}$ denote the entries of Z . Since Z is unimodular, the vectors

$$\xi'_i = \sum_{j=1}^{d_1} Z_{ij} \xi_j, \quad i = 1, \dots, d_1,$$

form a new integral basis ξ' of \mathfrak{t}_1 and

$$\nu \star \xi = \nu' \star \xi' \quad \text{with } \nu' = Z^{-T} \nu. \quad (18)$$

Since $(\omega, \nu') \cdot (p, q) = \omega \cdot p + \nu' \cdot Z^{-1}q$, the resonant lattice Λ' of $(\omega, \nu') \in \mathbb{R}^{k+d_1}$ has a basis formed by the l vectors $(\tilde{p}^i, Z\tilde{q}^i)$, $i = 1, \dots, l$. Let now $C_{ij} \in \mathbb{Z}$ be the entries of C . Since C is unimodular, another basis of Λ' is formed by the l vectors

$$\sum_{j=1}^l C_{ij}(\tilde{p}^j, Z\tilde{q}^j) = \left(\sum_{j=1}^l C_{ij}\tilde{p}^j, \sum_{j=1}^l C_{ij}Z\tilde{q}^j \right), \quad i = 1, \dots, l.$$

This is the basis (16), with $p^i = \sum_{j=1}^l C_{ij}\tilde{p}^j$ for $i = 1, \dots, l$, because the first l rows of (17) read $\sum_{j=1}^l C_{ij}Z\tilde{q}^j = r_i e^i$, $i = 1, \dots, l$. \square

Note that the resonance relations satisfied by the new frequency vector $(\omega, \nu') \in \mathbb{R}^{k+d_1}$ that correspond to the vectors of the basis (16) are

$$p^i \cdot \omega + r_i \nu'_i = 0, \quad i = 1, \dots, l. \quad (19)$$

Moreover, Lemma 5 gives a decomposition of the Lie algebra \mathfrak{t}_1 as a sum of two subalgebras, $\mathfrak{t}_1 = \mathfrak{t}_{\text{res}} \oplus \mathfrak{t}_0$, with $\mathfrak{t}_{\text{res}} = \text{Span}(\xi'_1, \dots, \xi'_l)$ of dimension l and $\mathfrak{t}_0 = \text{Span}(\xi'_{l+1}, \dots, \xi'_{d_1})$ of dimension $d_0 = d_1 - l$. This decomposition is such that if $\nu'' \in \mathbb{R}^{d_0}$ denotes the vector of the components of the frequency vector $\nu' \star \xi'$ in \mathfrak{t}_0 , that is $\nu'' = (\nu'_{l+1}, \dots, \nu'_{d_1})$, then the vector $(\omega, \nu'') \in \mathbb{R}^{k+d_0}$ is nonresonant.

We now proceed as in the proof of Theorem 1, but instead of the lifts S_1, \dots, S_k considered there we consider the lifts

$$S'_i = r S_i, \quad i = 1, \dots, k,$$

where $r = \max(r_1, \dots, r_l)$ (see Lemma 5). Thus $\Phi_1^{S'_i}(\overline{m}) = \Phi_r^{S_i}(\overline{m}) = \overline{\gamma}_i^r \cdot \overline{m}$. Correspondingly, we define

$$\omega' = \frac{\omega}{r}$$

so that $X = \sum_{j=1}^k \omega'_j S'_j$ and $\Phi_t^X = \Phi_{t\omega'}^{\{S'\}}$. A possible choice of logarithms of the powers $\bar{\gamma}_i^r$ of the phases would be $r\eta_i$, but we choose instead

$$\eta'_i = r\eta_i + \delta_i \quad \text{with} \quad \delta_i = \sum_{j=1}^l \frac{r}{r_j} p_i^j \xi'_j, \quad i = 1, \dots, k; \quad (20)$$

we will write $\delta = (\delta_1, \dots, \delta_k)$. The need of correcting the choice of logarithms with the elements δ_i will become clear in the lemma immediately below: these elements of the Lie algebra are precisely the corrections needed to make the map j' a covering map. Note that $\delta_i \in \mathfrak{t}_{\text{res}}$ and $\exp \delta_i = e_G$ (so that $\exp \eta'_i = \bar{\gamma}_i^r$) because the δ_i and η_i commute and the p_i^j and r/r_i are integers.

Lemma 6. *The map*

$$j' : \mathbb{T}^k \times G \rightarrow \mathcal{R}, \quad j'(\alpha, g) = g \exp(-\alpha \star \eta') \cdot \Phi_{\alpha}^{\{S'\}}(\bar{m}),$$

is a smooth $r^k : 1$ covering map.

Proof. The same arguments used in the proof of Lemma 2 show that j' is well defined, is surjective and is a local diffeomorphism at each point. Because of the latter property, if the cardinality of its fibers is constant then it is a covering map. Assume $j'(\alpha, g) = j'(\beta, h)$, or

$$\Phi_{\beta-\alpha}^{\{S'\}}(\bar{m}) = \exp(\beta \star \eta') h^{-1} g \exp(-\alpha \star \eta') \cdot \bar{m}. \quad (21)$$

This implies $\Phi_{\beta-\alpha}^{\{S'\}}(\bar{m}) \in G \cdot \bar{m}$. Since the S'_i have lifted period $1/r$, this happens if and only if $\beta \equiv \alpha \pmod{\frac{1}{r}}$ or

$$\beta \equiv \alpha + \frac{1}{r} u \pmod{1}$$

with some $u = (u_1, \dots, u_k) \in \mathbb{Z}_r^k$, where $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z} \equiv \{0, \dots, r-1\}$, and in this case $\Phi_{\beta-\alpha}^{S'_i}(\bar{m}) = \Phi_{u_i}^{S'_i}(\bar{m}) = \bar{\gamma}_i^{u_i} \cdot \bar{m}$. So, equation (21) implies

$$\Phi_{\beta-\alpha}^{\{S'\}}(\bar{m}) = \bar{\gamma}_1^{u_1} \dots \bar{\gamma}_k^{u_k} \cdot \bar{m} \quad \text{for some } u \in \mathbb{Z}_r^k.$$

Thus, if equation (21) is satisfied, there is $u \in \mathbb{Z}_r^k$ such that $\bar{\gamma}_1^{u_1} \dots \bar{\gamma}_k^{u_k} \cdot \bar{m} = \exp(\beta \star \eta') h^{-1} g \exp(-\alpha \star \eta') \cdot \bar{m}$, or

$$h^{-1} g = \bar{\gamma}_1^{u_1} \dots \bar{\gamma}_k^{u_k} \exp\left(-\frac{u_1}{r} \eta'_1 - \dots - \frac{u_k}{r} \eta'_k\right) = \exp\left(-\frac{1}{r} u \star \delta\right),$$

with $u \star \delta = \sum_{i=1}^k u_i \delta_i$. In conclusion, $j'(\alpha, g) = j'(\beta, h)$ if and only if there exists $u \in \mathbb{Z}_r^k$ such that

$$\beta = \alpha + \frac{u}{r} \pmod{1}, \quad h = g \exp\left(\frac{1}{r} u \star \delta\right). \quad (22)$$

Thus, the cardinality of the fibers of j' is r^k . \square

By (22), the deck transformations of the covering j' of \mathcal{R} give an action Ψ of \mathbb{Z}_r^k on $\mathbb{T}^k \times G$ defined by

$$\Psi_u(\alpha, g) = \left(\alpha + \frac{u}{r}, g \exp\left(\frac{1}{r} u \star \delta\right)\right)$$

that is free and satisfies $j' \circ \Psi_u = j'$ for all $u \in \mathbb{Z}_r^k$ (we recall that the deck transformations are the maps of a covering onto itself that preserve the fibers [18]). Therefore:

Lemma 7. \mathcal{R} is diffeomorphic to $(\mathbb{T}^k \times G)/\mathbb{Z}_r^k$ (the quotient being relative to the action Ψ).

Now

$$\Phi_t^X(j'(\alpha, g)) = \Phi_{t\omega'}^{\{S'\}}\left(g \exp(-\alpha \star \eta') \cdot \Phi_{\alpha}^{\{S'\}}(\overline{m})\right) = j'(\alpha + \omega' t, g \exp(t\omega' \star \eta')) \quad (23)$$

and instead of the subgroup Γ_1 of the proof of Theorem 1 we are lead to consider the subgroup

$$\Gamma_0 := \{ \exp(t\omega' \star \eta') : t \in \mathbb{R} \}.$$

By (19) and (20), $\sum_{i=1}^k \omega'_i \delta_i = \sum_{j=1}^l \frac{\omega \cdot p^j}{r_j} \xi'_j = - \sum_{j=1}^l \nu'_j \xi'_j$ and hence, using (14) and (18),

$$\omega' \star \eta' = \omega \star \eta + \sum_{i=1}^k \omega'_i \delta_i = \nu' \star \xi' - \sum_{j=1}^l \nu'_j \xi'_j = \sum_{j=l+1}^{d_1} \nu'_j \xi'_j.$$

Therefore, the closure of Γ_0 is the d_0 -dimensional torus $T_0 = \exp(\mathfrak{t}_0)$. For shortness, we write

$$\xi''_i = \xi'_{l+i}, \quad \nu''_i = \nu'_{l+i}, \quad i = 1, \dots, d_0,$$

and $\xi'' = (\xi''_1, \dots, \xi''_{d_0})$, so that $\omega' \star \eta' = \nu'' \star \xi''$. Thus, (23) becomes

$$\Phi_t^X(j'(\alpha, g)) = j'(\alpha + \omega' t, g \exp(t\nu'' \star \xi'')) \quad (24)$$

and the sets

$$\mathcal{T}'_{g, \overline{m}} := j'(\mathbb{T}^k \times gT_0), \quad g \in G,$$

are invariant under the flow of X and foliate \mathcal{R} . Consider the subgroup K of \mathbb{Z}_r^k defined as

$$K = \left\{ u \in \mathbb{Z}_r^k : \frac{u \cdot p^j}{r_j} \in \mathbb{Z} \text{ for all } j = 1, \dots, l \right\}. \quad (25)$$

Lemma 8. For each $g \in G$, $\mathcal{T}'_{g, \overline{m}}$ is diffeomorphic to $(\mathbb{T}^k \times gT_0)/K$ (the quotient being relative to the action Ψ).

Proof. Clearly, it suffices to consider $g = e_G$. It follows from Lemma 7 that $\mathcal{T}'_{\overline{m}}$ is diffeomorphic to the quotient of $\mathbb{T}^k \times T_0$ by the subgroup of the deck transformations that map $\mathbb{T}^k \times T_0$ to itself. Thus, we only have to show that this subgroup is K . Assume $(\alpha, h) \in \mathbb{T}^k \times T_0$. Since $\delta_i \in \mathfrak{t}_{\text{res}}$ and $T_0 = \exp(\mathfrak{t}_0)$, $\Psi_u(\alpha, h) = (\alpha + \frac{u}{r}, h \exp \frac{u \star \delta}{r})$ is in $\mathbb{T}^k \times T_0$ if and only if $\exp \frac{u \star \delta}{r} = e_G$. Since $\frac{u \star \delta}{r} = \sum_{i=1}^k \frac{u_i \delta_i}{r} = \sum_{j=1}^l \frac{u \cdot p^j}{r_j} \xi'_j$ and the ξ'_j are integral Lie algebra elements, this happens if and only if $\frac{u \cdot p^j}{r_j} \in \mathbb{Z}$ for all j , or $u \in K$. \square

Thus, the restriction of j' to $\mathbb{T}^k \times gT_0$ is a covering of $\mathcal{T}'_{g, \overline{m}}$ with fibers

$$\Psi_K(\alpha, h) = \{(\alpha + r^{-1}u, h) : u \in K\}, \quad (\alpha, h) \in \mathbb{T}^k \times gT_0.$$

(Note however that the preimage under j' of $\mathcal{T}'_{g, \overline{m}}$ is $\mathbb{T}^k \times gT_0 F_0$, where $F_0 = \{\exp(r^{-1}u \star \delta) : u \in \mathbb{Z}_r^k\}$). We thus introduce coordinates on these covering tori, with the map

$$i'_g : \mathbb{T}^k \times \mathbb{T}^{d_0} \rightarrow \mathcal{T}'_{g, \overline{m}}, \quad i'_g(\alpha, \beta) = j'(\alpha, g \exp(\beta \star \xi''))$$

which clearly is still a covering map, with fibers diffeomorphic to those of j' (see below). From (24) it then follows that

Lemma 9. For each $g \in G$, $i'_g : \mathbb{T}^k \times \mathbb{T}^{d_0} \rightarrow \mathcal{T}'_{g,\overline{m}}$ relates the linear flow on $\mathbb{T}^k \times \mathbb{T}^{d_0}$ with frequencies (ω', ν'') to the flow of X on $\mathcal{T}'_{g,\overline{m}}$.

As already remarked, the frequency vector (ω', ν'') is nonresonant.

Lemma 10. The sets $\mathcal{T}'_{g,\overline{m}}$ are the fibers of a fibration of \mathcal{R} whose base is diffeomorphic to $G/(T_0 F_0)$, with $F_0 = \{\exp(r^{-1}u \star \delta) : u \in \mathbb{Z}_r^k\} \equiv \mathbb{Z}_r^k/K$.

Proof. The quotient of \mathcal{R} by the sets $\mathcal{T}'_{g,\overline{m}}$ is diffeomorphic to the quotient of $\mathbb{T}^k \times G$ by the preimages under j' of such sets. The preimage of $\mathcal{T}'_{g,\overline{m}}$ is $\mathbb{T}^k \times gT_0 F_0$. \square

This proves Theorem 2 with $T = T_0 F_0$. Since T_0 and F_0 are abelian subgroups contained in T_1 , and their intersection is e_G , $T_0 F_0$ is abelian and diffeomorphic to $T_0 \times F_0$.

Remarks: 1. Our treatment of resonances differs from that in [8], which is restricted to the case $k = 1$. Following that approach, in the proof of Theorem 1 we would have taken T_2 as the smallest compact subgroup of G that contains the closure of Γ_2 (see also Remark 3 at the end of Section 4.2). If T_2 is connected, and hence a torus, when $k = 1$ there is nothing else to do. If T_2 is not connected, then it is the product of a torus T and of a finite group F , say of order r_1 . When $k = 1$, multiplying by r_1 the unique lift S_1 yields the power $\overline{\gamma}_1^{r_1}$ of the unique phase $\overline{\gamma}_1$ and automatically eliminates all resonances between the internal frequency ω_1 and the external frequencies ν . When $k \geq 2$, however, this procedure does not eliminate the possibility of resonances between the internal and the external frequencies.

2. The description of motions given in statement i. of Theorem 2 takes place not in the torus $\mathcal{T}'_m \subset \mathcal{R}$, but in a torus $\mathbb{T}^k \times gT_0$ that covers it. The frequencies (ω', ν'') are relative to the motion in such longer torus. In order to describe the motion in the tori \mathcal{T}'_m one should introduce angles on them. The relation between the chosen angles in $\mathbb{T}^k \times gT_0$ and those in given \mathcal{T}'_m can be obtained writing the group of deck transformations K defined in (25) in a normal form. This requires the computation of a unimodular, integer, $k \times k$ matrix M which makes $K = s_1 \mathbb{Z}_r \times \cdots \times s_l \mathbb{Z}_r \times \mathbb{Z}_r \times \cdots \times \mathbb{Z}_r$. It hence follows that the frequencies of the motion in \mathcal{T}'_m are $(\mu, \nu'') \in \mathbb{R}^{k+d_0}$ with $\mu = DM\omega'$, where D is the diagonal $k \times k$ matrix whose entries are $s_1, \dots, s_l, 1, \dots, 1$, $l = d - d_0$, and each s_i divides the integer r_i introduced in the proof of Theorem 2 (Lemma 5).

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